OPEN ACCESS
Reviews in Mathematical Physics
(2024) 2450054 (12 pages)

DOI: 10.1142/S0129055X24500545

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Study of symmetries through the action on torsors of the Janus symplectic group

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Received 22 August 2024
Accepted 30 October 2024
Published

In this paper, we focus on the Janus symplectic group. We explore its various symmetries and its action on the elements of the dual of its Lie algebra, called torsors. Special attention is given to the charge symmetry, which highlights the matter–antimatter duality within both sets of components.

Keywords: Dynamic groups; symplectic groups; Lorentz group, Poincaré group; Janus group; torsors of a Lie group; action on the torsors.

Mathematics Subject Classification 2020: 20C35, 22E70, 53D55, 83A04, 70H33

1. Introduction

The application of the coadjoint action of a symplectic group on the dual of its Lie algebra, initiated by the mathematician Jean-Marie Souriau, has shed light on specific aspects of the approach followed by physics. The orbit method is due to Kirillov ([4–7, 12, 13, 15, 21, 24, 25]).

Thus, the restricted Lorentz symplectic group, limited to its two orthochrone components, translates, through the invariance properties that result from it, the aspects of special relativity. In 1970, Souriau established that the analysis of the components of its moment makes it possible to shed light on the geometric nature

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of a spin (not quantized): see [19, 20]. He uses for this purpose symplectic methods ([8, 10, 22, 23]). In the theory of symplectic groups, we show a classification in terms of movements.

By operating the product of this group by that of the spatio-temporal translations, we obtain the restricted Poincaré symplectic group, always limited to its two orthochrone components. In its moment, we first find the energy related to the subgroup of temporal translations. Then the momentum, linked to the spatial translations, the two being linked by the invariance of the modulus of the energy-momentum four-vector under the action of the Lorentz group.

By adding a translation along a fifth dimension to the restricted Poincaré group, we form a Lie group to which we will give the name *Restricted Kaluza Group* ([1–3, 11, 14]). This group is not the 15-dimensional Kaluza group associated with a 5-dimensional Lorentzian manifold but a new 11-dimensional group, including 5-dimensional space-time translation. This new dimension endows the momentum with an additional scalar that can be identified with the electric charge q, which may be positive, negative, or zero, and is still not quantized. We then bring out the geometric translation according to a scalar ϕ due to endowing the masses with an invariant electric charge. Then, by bringing in a new symmetry reflecting the inversion of the fifth dimension, synonymous with an inversion of the scalar from q to $-q$, we double the number of its connected components from 2 to 4. The action on the moment then links this new symmetry to the inversion of the electric charge q. We thus deduce the geometric modeling of charge conjugation or **C**-*Symmetry*.[a] It's then logical to name this new extension, the *Restricted Janus Group*.

By introducing a new symmetry to the previous group, which we describe as **T**-*Symmetry*,[b] we build the *Janus Symplectic Group*. Thus, we double the number of connected components from four to eight, grouped into two subsets: "*Orthochronous*", conserving time and energy properties, and "*Antichronous*", reversing time and energy. Therefore, we bring forth the geometric translation of endowing masses with an invariant electric charge. As the Jean-Marie Souriau demonstrated as early as 1970, a pioneer in the theory of symplectic groups ([9, 19, 20]), this approach has allowed key elements, which have marked the progress of relativistic physics, to be given a purely geometric nature.

In relation to the world of physics, wouldn't the role of mathematics be to illuminate the path traveled? Conversely, could it be possible that the exploration of new symmetries, accompanying this decoding using symplectic groups, contains more than what we thought we put into it? That it could designate new paths to follow?

This is what we will consider with the Janus Symplectic Group with charge symmetry, by integrating the antichronous components of the Lorentz group, resulting

[a]which translates the matter–antimatter symmetry introduced by Dirac.
[b]which converts matter into antimatter with negative mass, a concept we could name *antimatter in the Feynman sense*.

from its simple axiomatic definition, with the obvious repercussions on the Poincaré group and its extensions.

2. Janus Symplectic Group

Let $\tilde{\mathbf{T}} := I_{1,3}$, $\tilde{\mathbf{P}} := -\tilde{\mathbf{T}}$ and

$$\forall\, \lambda, \nu \in \{0,1\}, \quad \mathcal{L}or(\tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda}) := \{L_n\tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda},\ L_n \in \mathcal{L}or_n\}.$$

Then, there are four connected components of $\mathcal{L}or$, given by[c]

$$\mathcal{L}or_n = \mathcal{L}or(\tilde{\mathbf{P}}^{0}\tilde{\mathbf{T}}^{0}), \quad \mathcal{L}or_s = \mathcal{L}or(\tilde{\mathbf{P}}^{1}\tilde{\mathbf{T}}^{0}),$$
$$\mathcal{L}or_t = \mathcal{L}or(\tilde{\mathbf{P}}^{0}\tilde{\mathbf{T}}^{1}), \quad \mathcal{L}or_{st} = \mathcal{L}or(\tilde{\mathbf{P}}^{1}\tilde{\mathbf{T}}^{1})$$

and we have the decomposition

$$\mathcal{L}or = \bigsqcup_{\nu,\lambda\in\{0,1\}} \mathcal{L}or(\tilde{\mathbf{P}}^{\nu}\tilde{\mathbf{T}}^{\lambda}). \tag{1}$$

Then, we define the Janus symplectic group.

Definition 2.1. The *Janus symplectic group* is defined as the subgroup of $\mathrm{GL}(6,\mathbb{R})$:

$$\mathcal{J}an := \left\{ \begin{pmatrix} L & 0 & D \\ 0 & (-1)^{\eta} & \phi \\ 0 & 0 & 1 \end{pmatrix},\ \eta \in \{0,1\} \ \wedge\ \phi \in \mathbb{R} \ \wedge\ L \in \mathcal{L}or \ \wedge\ D \in \mathbb{R}^4 \right\}.$$

The Janus symplectic group is therefore a subgroup of the group of isometries in dimension 5 given by[d]:

$$\mathrm{Aff}(\mathcal{O}(1,4)) := \left\{ \begin{pmatrix} L & D' \\ 0 & 1 \end{pmatrix},\ L \in \mathcal{O}(1,4) \ \wedge\ D' \in \mathbb{R}^5 \right\}$$

with $\tau_{1,4}(L) := I_{1,4}L^T I_{1,4}$ and $\mathcal{O}(1,4) := \{L \in \mathrm{GL}(5,\mathbb{R}),\ \tau_{1,4}(L)L = I_5\}$. The elements of $\mathrm{Aff}(\mathcal{O}(1,4))$ are the elements which preserve the distance between two events (*pentavectors*) $X := (t,x,y,z,\xi)$ and $X' := (t',x',y',z',\xi')$ given by

$$d(X,X') := c^2(t-t')^2 - (x-x')^2 - (y-y')^2 - (z-z')^2 - (\xi-\xi')^2.$$

[c]Equalities are shown by double inclusion. For example, let's demonstrate that $\mathcal{L}or_s = \mathcal{L}or(\tilde{\mathbf{P}}^1\tilde{\mathbf{T}}^0)$. Take $L \in \mathcal{L}or_s$ ($\det(L) = -1$ et $[L]_{00} \geq 1$). Then we have $\det(L\tilde{\mathbf{P}}) = -1$ and $[L\tilde{\mathbf{P}}]_{00} \geq 1$ i.e. we have $L_n := L\tilde{\mathbf{P}} \in \mathcal{L}or_n$. Since $\tilde{\mathbf{P}}^{-1} = \tilde{\mathbf{P}}$, we can conclude that $L = L_n\tilde{\mathbf{P}} \in \mathcal{L}or(\tilde{\mathbf{P}}^1\tilde{\mathbf{T}}^0)$. The inclusion in the other direction is trivial.

[d]$\mathrm{Aff}(\mathcal{O}(1,4))$ is the affine group associated with $\mathcal{O}(1,4)$, it is also defined by the semi-direct product $\mathrm{Aff}(\mathcal{O}(1,4)) := \mathcal{O}(1,4) \ltimes \mathbb{R}^5$. We can also define the symplectic Janus group as being the affine group associated with the subgroup of $\mathcal{O}(1,4)$ given by

$$\mathcal{E}lec := \left\{ \begin{pmatrix} L & 0 \\ 0 & (-1)^{\eta} \end{pmatrix},\ \eta \in \{0,1\} \ \wedge\ L \in \mathcal{L}or \right\}$$

called the *symplectic electric group* and we have $\mathcal{J}an := \mathrm{Aff}(\mathcal{E}lec)$.

This fifth dimension is of space type (we note the variable ξ). Each dimension is therefore associated with a symmetry, there are three types of symmetries:

- the $\mathbf{T}$-*symmetry*;
- the $\mathbf{P}_x$-*symmetry*, $\mathbf{P}_y$-*symmetry*, $\mathbf{P}_z$-*symmetry* grouped together what we call the $\mathbf{P}$-*symmetry*;
- the ξ-*symmetry* corresponding to the $\mathbf{C}$-*symmetry* (the charge conjugation).

This space of dimension 5 is a Minkowski metric space to which we have added one dimension, it has the metric $I_{1,4}$.

We also define the *restricted Janus group* is the subgroup of $\mathcal{J}an$ given by

$$\mathcal{J}an_n := \left\{ \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}, \ \phi \in \mathbb{R} \ \wedge \ L_n \in \mathcal{L}or_n \ \wedge \ D \in \mathbb{R}^4 \right\}.$$

Let

$$\mathbf{C} := \begin{pmatrix} I_4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} := \begin{pmatrix} \tilde{\mathbf{P}} & 0 \\ 0 & I_2 \end{pmatrix}, \quad \mathbf{T} := \begin{pmatrix} \tilde{\mathbf{T}} & 0 \\ 0 & I_2 \end{pmatrix}.$$

We have

$$\forall \lambda, \eta, \nu \in \{0,1\}, \quad \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix} \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda = \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore by Eq. (1):

$$\mathcal{J}an = \left\{ \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}, \ \lambda, \eta, \nu \in \{0,1\} \ \wedge \ \phi \in \mathbb{R} \ \wedge \ L_n \in \mathcal{L}or_n \ \wedge \ D \in \mathbb{R}^4 \right\}.$$

Definition 2.2. (i) The ***CPT****-group* is the subgroup $\mathcal{K}$ of $\mathcal{J}an$ of order 8 generated by $\mathbf{C}$, $\mathbf{P}$ and $\mathbf{T}$, i.e.

$$\mathcal{K} := \{\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda, \ \eta, \nu, \lambda \in \{0,1\}\} = \{I_6, \mathbf{T}, \mathbf{P}, \mathbf{PT}, \mathbf{C}, \mathbf{CT}, \mathbf{CP}, \mathbf{CPT}\}.$$

(ii) For all $\mathbf{X} \in \mathcal{K}$, the $\boldsymbol{X}$*-component of* $\mathcal{J}an$ is

$$\mathcal{J}an(\mathbf{X}) := \{JX, J \in \mathcal{J}an_n\}.$$

Thus, we have

$$\mathcal{J}an(\mathbf{C}^\eta\mathbf{P}^\nu\mathbf{T}^\lambda) = \left\{\begin{pmatrix} L_n\tilde{\mathbf{P}}^\nu\tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1\end{pmatrix},\ \phi\in\mathbb{R}\ \wedge\ L_n\in\mathcal{L}or_n\ \wedge\ D\in\mathbb{R}^4\right\}.$$

These eight components are the eight connected components of $\mathcal{J}an$, we have the decomposition:

$$\mathcal{J}an = \bigsqcup_{\mathbf{X}\in\mathcal{K}}\mathcal{J}an(\mathbf{X}) = \bigsqcup_{\eta,\nu,\lambda\in\{0,1\}}\mathcal{J}an(\mathbf{C}^\eta\mathbf{P}^\nu\mathbf{T}^\lambda).$$

The group $\mathcal{L}or$ is the Lie group of dimension 6 and its Lie algebra is

$$\mathfrak{lor} := \mathcal{A}(1,3) := \{\Lambda\in\mathcal{M}(4,\mathbb{R}), \tau_{1,3}(\Lambda) = -\Lambda\}.$$

Then, the group $\mathcal{J}an$ is a Lie group of dimension 11 and its Lie algebra is

$$\mathfrak{jan} = \left\{\begin{pmatrix}\Lambda & 0 & \Gamma\\ 0 & 0 & \varepsilon\\ 0 & 0 & 0\end{pmatrix}, \Lambda\in\mathcal{A}(1,3)\ \wedge\ \Gamma\in\mathbb{R}^4\ \wedge\ \varepsilon\in\mathbb{R}\right\}.$$

We have two characterizations[e]:

$$\left(\mathbb{R}^5\right)^* = \left\{\begin{pmatrix}\Gamma\\ \varepsilon\end{pmatrix}\mapsto -(P^T\ q)I_{1,4}\begin{pmatrix}\Gamma\\ \varepsilon\end{pmatrix} = -\tau(P)\Gamma - q\varepsilon,\ \begin{pmatrix}P\\ q\end{pmatrix}\in\mathbb{R}^5\right\},$$
$$\mathcal{A}(1,3)^* = \left\{\Lambda\mapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda),\ M\in\mathcal{A}(1,3)\right\}.$$

Then, we have[f]

$$\mathfrak{jan}^* = \left\{\{M\,|\,P\,|\,q\} : \begin{pmatrix}\Lambda & 0 & \Gamma\\ 0 & 0 & \varepsilon\\ 0 & 0 & 0\end{pmatrix}\mapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) - \tau(P)\Gamma - q\varepsilon,\ M\in\mathcal{A}(1,3)\ \wedge\ P\in\mathbb{R}^4\ \wedge\ q\in\mathbb{R}\right\}.$$

The *action of the group* $\mathcal{J}an$ *on* $\mathfrak{jan}^*$ is defined by the coadjoint representation i.e. for any $a\in\mathcal{J}an$ and any $\mu\in\mathfrak{jan}^*$, we denote this action by

$$a\bullet\mu := \mathrm{Ad}^*_a(\mu).$$

with

$$\mathrm{Ad}^* : \mathcal{J}an \to \mathrm{Aut}(\mathfrak{jan}^*)$$
$$a\mapsto \mathrm{Ad}^*_a : \mu\mapsto (Z\mapsto \mu(a^{-1}Za)).$$

[e]For all $\beta\in\mathbb{R}^*$, the application Φ_β which to $M\in\mathcal{A}(1,3)$ associates the linear form $\Lambda\mapsto \beta\mathrm{Tr}(M\Lambda)$ is an isomorphism of $\mathcal{A}(1,3)$ to $\mathcal{A}(1,3)^*$. Taking $\{A_{kl} := -E_{kl} + [I_{1,3}]_{ll}[I_{1,3}]_{kk}E_{lk},\ k,l\in\{1,\dots,4\},\ k<l\}$ the canonical basis of $\mathcal{A}(1,3)$, we have $\Phi_{-1/2}(A_{kl})(A_{kl}) = 1$, hence the choice of $\beta := -1/2$.

[f]The elements of $\mathfrak{jan}^*$ are called *torsors*.

Proposition 2.1. *Let*

$$a := \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{J}an, \quad \{M \,|\, P \,|\, q\} \in \mathfrak{jan}^*.$$

We have

$$a \bullet \{M \,|\, P \,|\, q\} = \{LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \,|\, LP \,|\, (-1)^\eta q\}.$$

Proof. We have

$$\begin{aligned}
&(a \bullet \{M \,|\, P \,|\, q\}) \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad = \{M \,|\, P \,|\, q\} \left(a^{-1} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} a \right) \\
&\quad = \{M \,|\, P \,|\, q\} \left(\begin{pmatrix} \tau(L) & 0 & -\tau(L)D \\ 0 & (-1)^\eta & (-1)^{\eta+1}\phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&\quad = \{M \,|\, P \,|\, q\} \begin{pmatrix} \tau(L)\Lambda L & 0 & \tau(L)(\Lambda D + \Gamma) \\ 0 & 0 & (-1)^\eta \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad = -\frac{1}{2}\mathrm{Tr}(M\tau(L)\Lambda L) - \tau(P)\tau(L)(\Lambda D + \Gamma) - (-1)^\eta q\varepsilon \\
&\quad = -\frac{1}{2}\mathrm{Tr}[(LM\tau(L) + 2D\tau(P)\tau(L))\Lambda] - \tau(LP)\Gamma - (-1)^\eta q\varepsilon \\
&\quad = -\frac{1}{2}\mathrm{Tr}[(LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D))\Lambda] - \tau(LP)\Gamma - (-1)^\eta q\varepsilon \\
&\quad = \{LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \,|\, LP \,|\, (-1)^\eta q\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

□

To describe the Lie algebra of $\mathcal{J}an$, we can also use the isomorphism of Lie algebras[g]:

$$\begin{aligned} j : (\mathbb{R}^3, \wedge) &\to (\mathcal{A}(3), [\ ,\ ]) \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}. \end{aligned}$$

[g] We have for all $u, v \in \mathbb{R}^3$: $u \wedge v = j(u)(v)$ and $j(u \wedge v) = [j(u), j(v)] = j(u)j(v) - j(v)j(u)$.

with $\wedge$ the cross product on $\mathbb{R}^3$ and $\mathcal{A}(3)$ the vector space of antisymmetric matrices of size 3. Then, we have

$$\mathfrak{jan} = \left\{ \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}, \ \Lambda \in \mathcal{A}(1,3) \ \wedge \ \Gamma \in \mathbb{R}^4 \ \wedge \ \varepsilon \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \beta, w, \gamma \in \mathbb{R}^3 \ \wedge \ v, \varepsilon \in \mathbb{R} \right\}.$$

Therefore, for all $\{M \,|\, P \,|\, q\} \in \mathfrak{jan}^*$, there are unique $\ell, g, p \in \mathbb{R}^3$ and $E, q \in \mathbb{R}$ such as

$$\{M \,|\, P \,|\, q\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} E \\ p \end{pmatrix} \middle| q \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -\frac{1}{2} \mathrm{Tr} \left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \right) - (E \ p^T) I_{1,3} \begin{pmatrix} v \\ \gamma \end{pmatrix} - q\varepsilon$$

$$= \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon.$$

We denote this last equality as

$$\{\ell \,|\, g \,|\, p \,|\, E \,|\, q\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The dual $\mathfrak{jan}^*$ has the following descriptions:

$$\left\{ \{\ell \,|\, g \,|\, p \,|\, E \,|\, q\} : \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon, \quad \ell, g, p \in \mathbb{R}^3 \ \wedge \ E, q \in \mathbb{R} \right\}.$$

Definition 2.3. Let

$$\mu := \{M \,|\, P \,|\, q\} := \{l \,|\, g \,|\, p \,|\, E \,|\, q\} \in \mathfrak{jan}^*$$

with relations

$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix}, \quad P = \begin{pmatrix} E \\ p \end{pmatrix}.$$

(i) The matrix $M := M(\mu) \in \mathcal{A}(1,3)$ is called the *moment matrix associated with* μ. The vector $\ell := \ell(\mu) \in \mathbb{R}^3$ is called the *angular momentum of* M, and the vector $g := g(\mu) \in \mathbb{R}^3$ is the *relativist barycenter of* M.

(ii) (a) The vector $P := P(\mu) \in \mathbb{R}^4$ is called the *stress-energy vector associated with* μ. The vector $p := p(\mu) \in \mathbb{R}^3$ is called the *linear momentum of* P, and the scalar $E := E(\mu) \in \mathbb{R}$ is called the *energy of* P.

(b) The *first Casimir number associated with* μ is defined by

$$C_1 := C_1(\mu) := P^T I_{1,3} P = E^2 - p^2.$$

(c) The *mass associated to* μ is defined by

$$m := m(\mu) := \operatorname{sign}(E)\sqrt{C_1} = \operatorname{sign}(E)\sqrt{E^2 - p^2}.$$

(iii) The scalar $q := q(\mu) \in \mathbb{R}$ is called the *electric charge associated with* μ.

We deduce a simple expression of the action of the **CPT**-group $\mathcal{K}$ on the torsors of $\mathfrak{jan}^*$.

Corollary 2.2. *Let* $\{l \,|\, g \,|\, p \,|\, E \,|\, q\} \in \mathfrak{jan}^*$. *For all* $\lambda, \eta, \nu \in \{0,1\}$, *we have*

$$(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{l \,|\, g \,|\, p \,|\, E \,|\, q\} = \{l \,|\, (-1)^{\lambda+\nu} g \,|\, (-1)^\nu p \,|\, (-1)^\lambda E \,|\, (-1)^\eta q\}.$$

Proof. We apply Proposition 2.1 with $a := \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda$:

$$\begin{aligned}
&(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{l \,|\, g \,|\, p \,|\, E \,|\, q\} \\
&\quad = (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} E \\ p \end{pmatrix} \middle| q \right\} \\
&\quad = \left\{ \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu \middle| I_{1,3} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \begin{pmatrix} E \\ p \end{pmatrix} \middle| (-1)^\eta q \right\} \\
&\quad = \left\{ \begin{pmatrix} 0 & (-1)^{\lambda+\nu} g^T \\ (-1)^{\lambda+\nu} g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} (-1)^\lambda E \\ (-1)^\nu p \end{pmatrix} \middle| (-1)^\eta q \right\} \\
&\quad = \{l \,|\, (-1)^{\lambda+\nu} g \,|\, (-1)^\nu p \,|\, (-1)^\lambda E \,|\, (-1)^\eta q\}.
\end{aligned}$$

□

So we have

$$\mathbf{C} \bullet \{l \,|\, g \,|\, p \,|\, E \,|\, q\} = \{l \,|\, g \,|\, p \,|\, E \,|\, -q\},$$

$$\mathbf{P} \bullet \{l\,|\,g\,|\,p\,|\,E\,|\,q\} = \{l\,|\,-g\,|\,-p\,|\,E\,|\,q\},$$
$$\mathbf{T} \bullet \{l\,|\,g\,|\,p\,|\,E\,|\,q\} = \{l\,|\,-g\,|\,p\,|\,-E\,|\,q\}.$$

Corollary 2.3. *Let* $\mu \in \mathfrak{jan}^*$. *For all* $\lambda, \eta, \nu \in \{0, 1\}$, *we have*

$$P((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu) = \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda P(\mu),$$
$$C_1((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu) = C_1(\mu),$$
$$m((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu) = (-1)^\lambda m(\mu).$$

Proof. Let $\mu := \{l\,|\,g\,|\,p\,|\,E\,|\,q\} \in \mathfrak{jan}^*$. We have for the stress-energy tensor

$$P(\mathbf{P} \bullet \mu) = P(\{l\,|\,-g\,|\,-p\,|\,E\,|\,q\}) = \begin{pmatrix} E \\ -p \end{pmatrix} = \tilde{\mathbf{P}} P(\mu),$$

		$\lambda = 0$	$\lambda = 1$
$\eta = 0$	$\nu = 0$	• neutral symetry • $a = I_6$ • $\mu' = \{\ell, g, p, E, q\}$	• $\mathbf{T}$ – symetry • $a = \mathbf{T}$ • $\mu' = \{\ell, -g, p, -E, q\}$
	$\nu = 1$	• $\mathbf{P}$ – symetry • $a = \mathbf{P}$ • $\mu' = \{\ell, -g, -p, E, q\}$	• $\mathbf{PT}$ – symetry • $a = \mathbf{PT}$ • $\mu' = \{\ell, g, -p, -E, q\}$
$\eta = 1$	$\nu = 0$	• $\mathbf{C}$ – symetry • $a = \mathbf{C}$ • $\mu' = \{\ell, g, p, E, -q\}$	• $\mathbf{CT}$ – symetry • $a = \mathbf{CT}$ • $\mu' = \{\ell, -g, p, -E, -q\}$
	$\nu = 1$	• $\mathbf{CP}$ – symetry • $a = \mathbf{CP}$ • $\mu' = \{\ell, -g, -p, E, -q\}$	• $\mathbf{CPT}$ – symetry • $a = \mathbf{CPT}$ • $\mu' = \{\ell, g, -p, -E, -q\}$

Fig. 1. This table lists the eight values of $\mu' := (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{l\,|\,g\,|\,p\,|\,E\,|\,q\}$ for all $\lambda, \eta, \nu \in \{0, 1\}$.

$$P(\mathbf{T} \bullet \mu) = P(\{l \,|\, -g \,|\, p \,|\, -E \,|\, q\}) = \begin{pmatrix} -E \\ p \end{pmatrix} = \tilde{\mathbf{T}} P(\mu),$$

$$P(\mathbf{P} \bullet \mu) = P(\{l \,|\, g \,|\, p \,|\, E \,|\, -q\}) = \begin{pmatrix} E \\ p \end{pmatrix} = P(\mu)$$

for the first Casimir number:

$$C_1((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu) = P(\mu)^T \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \tilde{\mathbf{P}}^\nu \tilde{\mathbf{P}}^\lambda P(\mu) = P(\mu)^T I_{1,3} P(\mu) = C_1(\mu)$$

for the mass:

$$\begin{aligned} m((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu) &= \operatorname{sign}(E((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu)) \sqrt{C_1((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu)} \\ &= \operatorname{sign}((-1)^\lambda E) \sqrt{C_1(\mu)} = (-1)^\lambda m(\mu). \qquad \square \end{aligned}$$

Therefore the elements variable by these actions are

$$P(\mathbf{P} \bullet \mu) = \tilde{\mathbf{P}} P(\mu), \quad P(\mathbf{T} \bullet \mu) = \tilde{\mathbf{T}} P(\mu), \quad m(\mathbf{T} \bullet \mu) = -m(\mu) \tag{2}$$

and we have the above table in Fig. 1.

3. Discussion and Conclusion

In this paper, we have performed a double extension of the restricted Poincaré group limited to its orthochronous components, which are classically used in physics. This extension also includes the transition from the four-dimensional Minkowski space-time to a new space of the same dimension, to which we have added a translation along an additional fifth dimension to form a new Lie group. The existence of this additional subgroup results in the invariance of a scalar, identified as the electric charge. A symmetry is introduced along this fifth dimension, and we have shown that this leads to the inversion of the electric charge. This provides a geometric representation of the symmetry between matter and antimatter.

In 1905, Albert Einstein revolutionized physics by introducing the theory of special relativity, integrating time, via the constant c, as a coordinate comparable to spatial dimensions in the geometry of Minkowski space. Ten years later, with his field equation, he explained phenomena such as the precession of Mercury's perihelion and the deflection of light, while laying the foundations of modern cosmology. The Big Bang theory, supported by Edwin Hubble's observations and Friedmann's work, revealed that the early universe was characterized by extreme density and temperature conditions.

Simultaneously, quantum mechanics emerged with the discovery of antimatter by Paul Dirac, an essential postulate in the cosmological model where, a fraction of a second after the Big Bang, matter and antimatter coexisted in equilibrium with gamma photons. However, the progressive disappearance of antimatter, implied by annihilations, remains a mystery, reinforced by the discovery of the cosmic microwave background in 1965.

The paradox of baryonic asymmetry found a theoretical response in 1967 with Sakharov, who postulated the existence of a twin universe symmetric to ours according to **CPT** symmetry, containing antimatter and evolving with an opposite arrow of time [16–18]. This model is based on a natural extension of fundamental symmetries, although the complete separation of the two universes leaves unresolved questions.

Within the framework of a geometric approach via dynamical groups, and building on the theory of symplectic groups by Souriau [20], we consider here an extension of Sakharov's model, the Janus model. The latter proposes an alternative vision where the two universes, far from being disjoint, are connected by a covering structure. This configuration allows gravitational interaction between particles of opposite masses, thereby redefining **CPT** symmetry in a larger geometric framework, including the antichronous components of the Poincaré group.

In conclusion, this approach not only sheds light on the path physics has taken but also suggests new avenues, offering a possible resolution to contemporary crises in cosmology, as evidenced by the data from the Hubble and James Webb telescopes. A more detailed study will be presented, including a system of coupled field equations modeling the gravitational interactions induced by these new symmetries.

Acknowledgment

Thanks to H. Zejli for correcting my many typos and improving the layout of my article. A modest task indeed, but so useful that my readers are not put off by the complexity of my scientific work.

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